METHODS OF REFINING THE CLASSICAL THEORY OF BENDING AND EXTENSION OF PLATES

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An account is given of very simple methods of refining the classical theory. These methods result from an asymptotic approach to the construction of the two-dimensional equations for the bending and extension of plates suggested in [1 and 2].

It was proved in [1 to 5] that the equations and boundary conditions in the classical theory are identical with the equations and boundary conditions for the zeroth approximation in the basic iterative process. Here it is shown that, in order to refine the state of stress at the edge of the plate (in the terms of the same order as the basic stresses in the classical theory), the state of stress determined in the classical theory must be supplemented by a state of stress due to the edge torsion and the plane deformation at the edge. It is also shown that, in order to refine the results given by the classical theory at points distant from the edge (the construction of an approximate theory for which the errors in the stresses have the order h^2, and not h, with respect to h^0), one must retain equations of the classical theory but certain alterations should be made in the boundary conditions. The form of the new boundary conditions for a free, a simply supported, and fully fixed edge will be formulated.

1. The middle surface of the plate will be referred to the curvilinear coordinates α, β. The coordinate γ will be measured from the middle surface along the normal to it. Use will be made of the auxiliary variables

$$\zeta = \frac{z}{h}, \qquad \xi = \frac{1}{h}\int_{\alpha_0}^{\alpha} \frac{d\alpha}{H_\alpha} \tag{1.1}$$

It is assumed that the edge of the plate corresponds with the coordinate line $\alpha = \alpha_0$ ($\xi = 0$) and that this line is smooth. As in [2], we will consider the symmetric problem (corresponding to extension) and the antisymmetric problem (corresponding to bending).

The conditions on the upper and lower surfaces of the plate have the form

for the antisymmetric problem

$$\sigma_{\gamma\gamma} = \pm {}^1/_2\, p(\alpha, \beta), \qquad \sigma_{\alpha\gamma} = {}^1/_2 h^{-1} p_\alpha(\alpha, \beta) \quad (\alpha\beta) \quad \text{when } \zeta = \pm 1$$

for the symmetric problem

$$\sigma_{\gamma\gamma} = {}^1/_2\, q(\alpha, \beta), \qquad \sigma_{\alpha\gamma} = \pm {}^1/_2\, h^{-1} q_\alpha(\alpha, \beta) \quad (\alpha\beta) \quad \text{when } \zeta = \pm 1$$

Here and in what follows the symbol $(\alpha\beta)$ will be used to denote the existence of a second relation derived from the given expression by changing α into β, and vice versa.

As has been shown in [1 and 2] and also (for the case of free boundaries) in [4 and 5], the asymptotic method of deriving the two-dimensional equations of the theory of thin elastic plates reduces to the construction of two forms of the state of stress (basic and boundary), each of which can be constructed with the aid of an appropriate iteration process. (In the foreign literature [4 and 5], the concepts of the interior problem and of the boundary-layer problem have been introduced). Moreover, if Q is any one of the stresses or displacements in the total state of stress in the plate, it can be expressed in the form

$$Q = h^{-q_1} \sum_{s=0}^{S} h^s Q^{(s)} + h^{-q_2} \sum_{s=0}^{S} h^s [Q_{(1)}{}^{(s)} + Q_{(2)}{}^{(s)}] \qquad (1.2)$$

The first term in (1.2) represent what has been called the basic state of stress in [1]. It extends throughout the whole plate and is determined with the aid of the basic iteration process. The index q_1 assumes the following values:

$$q_1 = 2 \quad \text{for } \sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\beta\beta}, \qquad q_1 = 1 \quad \text{for } \sigma_{\alpha\gamma}, \sigma_{\beta\gamma}$$
$$q_1 = 0 \quad \text{for } \sigma_{\gamma\gamma}, \qquad q_1 = 2 \text{ for } \quad u_\alpha, u_\beta \qquad (1.3)$$
$$q_1 = 1 \quad \text{for } W \quad \text{in the symmetrical problem}$$
$$q_1 = 3 \quad \text{for } W \quad \text{in the antisymmetrical problem}$$

The stresses and displacements in the basic state of stress can be represented in the form of polynomials in ζ, where the degrees of these polynomials increase as the order of approximation increases. For the first two approximations the dependence of the stresses and displacements on the variable ζ is determined by Formulas

in the symmetric case

$$W^{(i)} = \zeta w^{(i)}(\alpha, \beta), \qquad u_\alpha{}^{(i)} = v_\alpha{}^{(i)}(\alpha, \beta) \ (\alpha\beta), \qquad \sigma_{\alpha\alpha}{}^{(i)} = \tau_{\alpha\alpha}{}^{(i)}(\alpha, \beta) \ (\alpha\beta) \qquad (1.4)$$
$$\sigma_{\alpha\beta}{}^{(i)} = \tau_{\alpha\beta}{}^{(i)}(\alpha, \beta), \qquad \sigma_{\alpha\gamma}{}^{(i)} = \zeta\tau_{\alpha\gamma}{}^{(i)}(\alpha, \beta) \ (\alpha\beta), \quad \sigma_{\gamma\gamma}{}^{(i)} = S_{\gamma\gamma}{}^{(i)}(\alpha, \beta) + {}^1/_2\zeta^2\tau_{\gamma\gamma}{}^{(i)}(\alpha, \beta)$$
$$Ew^{(i)} = -\nu(\tau_{\alpha\alpha}{}^{(i)} + \tau_{\beta\beta}{}^{(i)})$$

in the antisymmetric case

$$W^{(i)} = w^{(i)}(\alpha, \beta), \qquad u_\alpha{}^{(i)} = \zeta v_\alpha{}^{(i)}(\alpha, \beta) \ (\alpha\beta), \qquad \sigma_{\alpha\alpha}{}^{(i)} = \zeta\tau_{\alpha\alpha}{}^{(i)}(\alpha, \beta) \ (\alpha\beta)$$
$$\sigma_{\alpha\beta}{}^{(i)} = \zeta\tau_{\alpha\beta}{}^{(i)}(\alpha, \beta), \qquad \sigma_{\alpha\gamma}{}^{(i)} = S_{\alpha\gamma}{}^{(i)}(\alpha, \beta) + \zeta^2\tau_{\alpha\gamma}{}^{(i)}(\alpha, \beta) \ (\alpha\beta) \qquad (1.5)$$
$$\sigma_{\gamma\gamma}{}^{(i)} = \zeta S_{\gamma\gamma}{}^{(i)}(\alpha, \beta) + \zeta^3\tau_{\gamma\gamma}{}^{(i)}(\alpha, \beta); \qquad v_\alpha{}^{(i)} = -H_\alpha \frac{\partial w^{(i)}}{\partial \alpha} \ (\alpha\beta)$$
$$W^{(2)} = w^{(2)}(\alpha, \beta) - \frac{\nu}{2E}\zeta^2(\tau_{\alpha\alpha}{}^{(0)} + \tau_{\beta\beta}{}^{(0)}) \quad (i=0, 1)$$

The equations that determine the coefficients of these polynomials are in each approximation analogous to the equations in the classical theory [2]. For the zeroth approximation they are identical with the equations in the classical theory and can be reduced to a nonhomogeneous biharmonic equation. In the first approximation, they are reduced to a homogeneous biharmonic equation. Thus, these coefficients are completely determinate, provided in each approximation one has two boundary conditions on the edge $\alpha = \alpha_0$.

The second term in (1.2) represents the boundary-layer state of stresses which quickly damps with distance from the edge, and can be determined with the aid of an auxiliary iteration process [2]. The index q_2 assumes the following values:

$$q_2 = 2 \quad \text{for } \sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\alpha\gamma}, \sigma_{\beta\beta}, \sigma_{\beta\gamma}, \sigma_{\gamma\gamma}, \quad q_2 = 1 \quad \text{for } u_\alpha, u_\beta, W \qquad (1.6)$$

The auxiliary iteration process consists in repeated integration of the systems of equations for the edge torsion and the plane deformation at the edge. These equations were derived in [2]. Utilizing (1.1) and taking the first two terms of the Taylor series for H_α, H_β and k_β at the point $\alpha = \alpha_0$, the above equations for $s = 0, 1, 2$ can be written in the form

$$\frac{\partial\bar\sigma_{\alpha\beta}^{(s)}}{\partial\xi} + \frac{\partial\bar\sigma_{\beta\gamma}^{(s)}}{\partial\zeta} = -H_{\beta0}\frac{\partial\bar\sigma_{\beta\beta}^{(s-1)}}{\partial\beta} + k_{\beta0}H_{\beta0}\xi\frac{\partial\bar\sigma_{\beta\beta}^{(s-2)}}{\partial\beta} - 2k_{\beta0}\bar\sigma_{\alpha\beta}^{(s-1)} + 2k_{\beta0}^2\xi\bar\sigma_{\alpha\beta}^{(s-2)}$$

$$E\frac{\partial\bar u_\beta^{(s)}}{\partial\xi} - 2(1+\nu)\sigma_{\alpha\beta}^{(s)} = -E\left[H_{\beta0}\frac{\partial\bar u_\alpha^{(s-1)}}{\partial\beta} - k_{\beta0}H_{\beta0}\xi\frac{\partial\bar u_\alpha^{(s-2)}}{\partial\beta} - k_{\beta0}\bar u_\beta^{(s-1)} + k_{\beta0}^2\xi\bar u_\beta^{(s-2)}\right]$$

$$E\frac{\partial\bar u_\beta^{(s)}}{\partial\zeta} - 2(1+\nu)\bar\sigma_{\beta\gamma}^{(s)} = -E\left[H_{\beta0}\frac{\partial\overline{W}^{(s-1)}}{\partial\beta} - k_{\beta0}H_{\beta0}\xi\frac{\partial\overline{W}^{(s-2)}}{\partial\beta}\right] \tag{1.7}$$

$$\frac{\partial\bar\sigma_{\alpha\alpha}^{(s)}}{\partial\xi} + \frac{\partial\bar\sigma_{\alpha\gamma}^{(s-1)}}{\partial\zeta} = -H_{\beta0}\frac{\partial\bar\sigma_{\alpha\beta}^{(s-1)}}{\partial\beta} + k_{\beta0}H_{\beta0}\xi\frac{\partial\bar\sigma_{\alpha\beta}^{(s-2)}}{\partial\beta} - k_{\beta0}(\bar\sigma_{\alpha\alpha}^{(s-1)} - \bar\sigma_{\beta\beta}^{(s-1)}) + k_{\beta0}^2\xi(\bar\sigma_{\alpha\alpha}^{(s-2)} - \bar\sigma_{\beta\beta}^{(s-2)})$$

$$\frac{\partial\bar\sigma_{\alpha\gamma}^{(s)}}{\partial\xi} + \frac{\partial\bar\sigma_{\gamma\gamma}^{(s)}}{\partial\zeta} = -H_{\beta0}\frac{\partial\bar\sigma_{\beta\gamma}^{(s-1)}}{\partial\beta} + k_{\beta0}H_{\beta0}\xi\frac{\partial\bar\sigma_{\beta\gamma}^{(s-2)}}{\partial\beta} - k_{\beta0}\bar\sigma_{\alpha\gamma}^{(s-1)} + k_{\beta0}^2\xi\bar\sigma_{\alpha\gamma}^{(s-2)}$$

$$E\frac{\partial\bar u_\alpha^{(s)}}{\partial\xi} - [\bar\sigma_{\alpha\alpha}^{(s)} - \nu(\bar\sigma_{\beta\beta}^{(s)} + \bar\sigma_{\gamma\gamma}^{(s)})] = 0 \tag{1.8}$$

$$E\frac{\partial W}{\partial\zeta} - [\bar\sigma_{\gamma\gamma}^{(s)} - \nu(\bar\sigma_{\alpha\alpha}^{(s)} + \bar\sigma_{\beta\beta}^{(s)})] = 0, \qquad E\left(\frac{\partial\overline{W}^{(s)}}{\partial\xi} + \frac{\partial u_\alpha^{(s)}}{\partial\zeta}\right) - 2(1+\nu)\bar\sigma_{\alpha\gamma}^{(s)} = 0$$

$$\sigma_{\beta\beta}^{(s)} - \nu(\sigma_{\alpha\alpha}^{(s)} + \sigma_{\gamma\gamma}^{(s)}) = E\left[H_{\beta0}\frac{\partial u_\beta^{(s-1)}}{\partial\beta} - k_{\beta0}H_{\beta0}\xi\frac{\partial\bar u_\beta^{(s-2)}}{\partial\beta} + k_{\beta0}u_\alpha^{(s-1)} - k_{\beta0}^2\xi u_\alpha^{(s-2)}\right]$$

where $H_{\beta0} = H_\beta|_{\alpha=\alpha_0}$ and $k_{\beta0} = k_\beta|_{\alpha=\alpha_0}$ is the curvature of the edge $\alpha = \alpha_0$. Any one of the stresses or displacements can be detrmined in the form

$$\overline{Q}^{(t)} = Q_{(1)}^{(t)} + Q_{(2)}^{(t)} \tag{1.9}$$

and it can be assumed that

$$\overline{Q}^{(t)} = Q_{(1)}^{(t)} = Q_{(2)}^{(t)} \equiv 0 \qquad \text{for } t < 0 \tag{1.10}$$

The quantities $Q_{(1)}^{(t)}$ and $Q_{(2)}^{(t)}$ can be determined so that (1.11)

$$\sigma_{\alpha\alpha(1)}^{(0)} = \sigma_{\alpha\gamma(1)}^{(0)} = \sigma_{\beta\beta(1)}^{(0)} = \sigma_{\gamma\gamma(1)}^{(0)} = u_{\alpha(1)}^{(0)} = W_{(1)}^{(0)} \equiv 0, \qquad \sigma_{\alpha\beta(2)}^{(0)} = \sigma_{\beta\gamma(2)}^{(0)} = u_{\beta(2)}^{(0)} \equiv 0$$

.n, taking all terms in (1.7) and (1.8) with subscripts (1) or (2), we obtain a system for the determination of $Q_{(1)}^{(s)}$ or $Q_{(2)}^{(s)}$, respectively.

The systems (1.7) and (1.8) lead to harmonic and biharmonic equations (with respect to the variables ξ, ζ), respectively. They are homogeneous when $s = 0$ and nonhomogeneous when $s > 0$. In the half-strip $-1 \leqslant \zeta \leqslant 1$, $\xi \leqslant 0$, one must construct for both systems solutions which are damped out as $\zeta \to -\infty$ and which satisfy the following conditions:

for (1.7)

$$\bar\sigma_{\beta\gamma}^{(s)} \equiv \sigma_{\beta\gamma(1)}^{(s)} + \sigma_{\beta\gamma(2)}^{(s)} = 0 \qquad \text{when } \zeta = \pm 1 \tag{1.12}$$

for (1.8)

$$\bar\sigma_{\alpha\gamma}^{(s)} \equiv \sigma_{\alpha\gamma(1)}^{(s)} + \sigma_{\alpha\gamma(2)}^{(s)} = 0, \qquad \bar\sigma_{\gamma\gamma}^{(s)} \equiv \sigma_{\gamma\gamma(1)}^{(s)} + \sigma_{\gamma\gamma(2)}^{(s)} = 0 \qquad \text{when } \zeta = \pm 1 \tag{1.13}$$

The conditions for the existence of damped solutions of systems (1.7) and

(1.8) can be written in the following form [2]

$$\int_{-1}^{1}\zeta\bar\sigma_{\alpha\alpha}^{(s)}\big|_{\xi=0}d\zeta=\int_{-\infty}^{0}d\xi\int_{-1}^{1}\Big\{\zeta\Big[-H_{\beta 0}\frac{\partial\bar\sigma_{\alpha\beta}^{(s-1)}}{\partial\beta}+k_{\beta 0}H_{\beta 0}\xi\frac{\partial\bar\sigma_{\alpha\beta}^{(s-2)}}{\partial\beta}-k_{\beta 0}(\bar\sigma_{\alpha\alpha}^{(s-1)}-\bar\sigma_{\beta\beta}^{(s-1)})+$$

$$+k_{\beta 0}^{2}\xi(\bar\sigma_{\alpha\beta}^{(s-2)}-\bar\sigma_{\beta\beta}^{(s-2)})\Big]-\xi\Big[-H_{\beta 0}\frac{\partial\bar\sigma_{\beta\gamma}^{(s-1)}}{\partial\beta}+k_{\beta 0}H_{\beta 0}\xi\frac{\partial\bar\sigma_{\beta\gamma}^{(s-2)}}{\partial\beta}-$$

$$-k_{\beta 0}\bar\sigma_{\alpha\gamma}^{(s-1)}+k_{\beta 0}^{2}\xi\bar\sigma_{\alpha\gamma}^{(s-2)}\Big]\Big\}d\zeta \qquad (1.14)$$

$$\int_{-1}^{1}\bar\sigma_{\alpha\gamma}^{(s)}\big|_{\xi=0}d\zeta=\int_{-\infty}^{0}d\xi\int_{-1}^{1}\Big[-H_{\beta 0}\frac{\partial\bar\sigma_{\beta\gamma}^{(s-1)}}{\partial\beta}+k_{\beta 0}H_{\beta 0}\xi\frac{\partial\bar\sigma_{\beta\gamma}^{(s-2)}}{\partial\beta}-k_{\beta 0}\bar\sigma_{\alpha\gamma}^{(s-1)}+k_{\beta 0}^{2}\xi\bar\sigma_{\alpha\gamma}^{(s-2)}\Big]d\zeta$$

$$\int_{-1}^{1}\bar\sigma_{\alpha\alpha}^{(s)}\big|_{\xi=0}d\zeta=\int_{-\infty}^{0}d\xi\int_{-1}^{1}\Big[-H_{\beta 0}\frac{\partial\bar\sigma_{\alpha\beta}^{(s-1)}}{\partial\beta}+k_{\beta 0}H_{\beta 0}\xi\frac{\partial\bar\sigma_{\alpha\beta}^{(s-2)}}{\partial\beta}-k_{\beta 0}(\bar\sigma_{\alpha\alpha}^{(s-1)}-\bar\sigma_{\beta\beta}^{(s-1)})+$$

$$+k_{\beta 0}^{2}\xi(\bar\sigma_{\alpha\alpha}^{(s-2)}-\bar\sigma_{\beta\beta}^{(s-2)})\Big]d\zeta \qquad (1.15)$$

$$\int_{-1}^{1}\bar\sigma_{\alpha\beta}^{(s)}\big|_{\xi=0}d\zeta=\int_{-\infty}^{0}d\xi\int_{-1}^{1}\Big[-H_{\beta 0}\frac{\partial\bar\sigma_{\beta\beta}^{(s-1)}}{\partial\beta}+k_{\beta 0}H_{\beta 0}\xi\frac{\partial\bar\sigma_{\beta\beta}^{(s-2)}}{\partial\beta}-2k_{\beta 0}\bar\sigma_{\alpha\beta}^{(s-1)}+2k_{\beta 0}^{2}\xi\bar\sigma_{\alpha\beta}^{(s-2)}\Big]d\zeta$$

The conditions (1.14) in the antisymmetric problem and conditions (1.15) in the symmetric problem yield two sequences of boundary conditions on the edge $\alpha = \alpha_0 (\xi = 0)$ for the coefficients of the expansion (1.2). The boundary conditions on the edge surface

$$\sigma_{\alpha\alpha}=\sigma_{\alpha\beta}=\sigma_{\alpha\gamma}=0,\qquad \sigma_{\alpha\alpha}=\sigma_{\alpha\beta}=W=0,\qquad u_\alpha=u_\beta=W=0\qquad \text{for } \alpha=\alpha_0$$

which correspond respectively to a free, simply supported, and fully fixed edge, each yield three sequences of boundary conditions for the coefficients in the expansions (1.2). They can be written in the form [2]

when $\alpha=\alpha_0\,(\xi=0)$

$$\sigma_{\alpha\alpha}^{(s)}+\sigma_{\alpha\alpha\,(1)}^{(s)}+\sigma_{\alpha\alpha\,(2)}^{(s)}=0,\quad \sigma_{\alpha\beta}^{(s)}+\sigma_{\alpha\beta\,(1)}^{(s)}+\sigma_{\alpha\beta\,(2)}^{(s)}=0,\quad \sigma_{\alpha\gamma}^{(s-1)}+\sigma_{\alpha\gamma\,(1)}^{(s)}+\sigma_{\alpha\gamma\,(2)}^{(s)}=0 \qquad (1.16)$$

$$\sigma_{\alpha\alpha}^{(s)}+\sigma_{\alpha\alpha\,(1)}^{(s)}+\sigma_{\alpha\alpha\,(2)}^{(s)}=0,\quad \sigma_{\alpha\beta}^{(s)}+\sigma_{\alpha\beta\,(1)}^{(s)}+\sigma_{\alpha\beta\,(2)}^{(s)}=0,\quad W^{(s)}+W_{(1)}^{(a)}+W_{(2)}^{(a)}=0 \qquad (1.17)$$

$$u_{\alpha}^{(s)}+u_{\alpha\,(1)}^{(s-1)}+u_{\alpha\,(2)}^{(s-1)}=0,\quad u_{\beta}^{(s)}+u_{\beta\,(1)}^{(s-1)}+u_{\beta\,(2)}^{(s-1)}=0,\quad W^{(s)}+W_{(1)}^{(a)}+W_{(2)}^{(a)}=0 \qquad (1.18)$$

where $a = s$ for the symmetric problem and $a = s - 2$ for the antisymmetric problem.

The arbitray constants of integration of the equations of the basic and auxiliary iteration processes must be used to satisfy the boundary conditions on $\alpha = \alpha_0$ $(\xi = 0)$. In [2] it was shown that these arbitrary constants are sufficient for satisfying not only conditions (1.16) but also conditions (1.14) and (1.15).

For the sake of simplicity we will use $B^{(s)}$ to denote the function determination of which gives the solution of the system of equations of the basic iteration process in any approximation . Similarly $\Psi^{(s)}$ and $\Phi^{(s)}$ will denote the functions for solution of systems (1.7) and (1.8), respectively.

The equations determining $B^{(s)}$, $\Psi^{(s)}$ and $\Phi^{(s)}$, are integrated separately. It is most important (i) to separate also the boundary value problems arising from the boundary conditions to be satisfied by the above functions, (ii) to fix a sequence of determinations of the functions $B^{(s)}$, $\Psi^{(s)}$ and $\Phi^{(s)}$ and (iii) to formulate the boundary conditions separately for each of these functions.

Below, we have considered this question for the functions $B^{(0)}$, $\Psi^{(0)}$, $\Phi^{(0)}$ and $B^{(1)}$, which suffices for the purposes of this paper. (The antisymmetric and symmetric problems have been treated separately).

N o t e . In the following, the notation $(1.16)_k$ means Equations (1.16) for $s = k$.

It is assumed that the quantities $Q_{(1)}^{(s)}$ and $Q_{(2)}^{(s)}$ vanish as $\xi \to -\infty$ faster than an arbitrary negative power of ξ .

2. In the antisymmetric problem with a free edge, the determination of the constants of integration in $B^{(0)}$, $B^{(1)}$, $\Psi^{(0)}$, $\Psi^{[1]}$, $\Phi^{(0)}$ and $\Phi^{(1)}$ requires the use of conditions $(1.16)_0$, $(1.16)_1$ and also conditions $(1.14)_0$, $(1.14)_1$ which, with the aid of (1.16), (1.19) and (1.11), can be transformed into the form

$$\int_{-1}^{1} \zeta \sigma_{\alpha\alpha}^{(1)}\Big|_{\alpha=\alpha_0} d\zeta = 0 \tag{2.1}$$

$$\int_{-1}^{1} \zeta \sigma_{\alpha\alpha}^{(1)}\Big|_{\alpha=\alpha_0} d\zeta = \int_{-\infty}^{0} d\xi \int_{-1}^{1} \left\{\zeta \left[H_{\beta 0} \frac{\partial \sigma_{\alpha\beta(1)}^{(0)}}{\partial\beta} + k_{\beta 0}\left(\sigma_{\alpha\alpha(2)}^{(0)} - \sigma_{\beta\beta(2)}^{(0)}\right)\right] - \right.$$
$$\left. - \xi\left[H_{\beta 0}\frac{\partial \sigma_{\beta\gamma(1)}^{(0)}}{\partial\beta} + k_{\beta 0}\sigma_{\alpha\gamma(2)}^{(0)}\right]\right\} d\zeta \tag{2.2}$$

$$\int_{-1}^{1} \sigma_{\alpha\gamma}^{(0)}\Big|_{\alpha=\alpha_0} d\zeta = \int_{-\infty}^{0} d\xi \int_{-1}^{1} \left[H_{\beta 0}\frac{\partial \sigma_{\beta\gamma(1)}^{(0)}}{\partial\beta} + k_{\beta 0}\sigma_{\alpha\gamma(2)}^{(0)}\right] d\zeta \tag{2.3}$$

$$\int_{-1}^{1} \sigma_{\alpha\gamma}^{(1)}\Big|_{\alpha=\alpha_0} d\zeta = \int_{-\infty}^{0} d\xi \int_{-1}^{1} \left[H_{\beta 0}\frac{\partial \bar{\sigma}_{\beta\gamma}^{(1)}}{\partial\beta} - k_{\beta 0}H_{\beta 0}\xi\frac{\partial \sigma_{\beta\gamma(1)}^{(0)}}{\partial\beta} + k_{\beta 0}\bar{\sigma}_{\alpha\gamma}^{(1)} - k_{\beta 0}^{2}\xi\sigma_{\alpha\gamma(2)}^{(0)}\right] d\zeta \tag{2.4}$$

Let us consider the auxiliary Problem 1. We will construct the solution $Q^{[1]}$ of the homogeneous system of Equations $(1.7)_0$ in the half-strip $-1 \leqslant \zeta \leqslant 1,\ \xi \leqslant 0,$ which satisfies the conditions

$$\sigma_{\beta\gamma}^{[1]}\Big|_{\zeta=\pm 1} = 0, \qquad Q^{[1]}\Big|_{\xi=-\infty} = 0, \qquad \sigma_{\alpha\beta}^{[1]}\Big|_{\xi=0} = -\zeta \tag{2.5}$$

The solution of this problem is easily found, for example, by separation of variables, in the form

$$\sigma_{\alpha\beta}^{[1]} = \partial\Psi^{[1]}/\partial\xi, \qquad \sigma_{\beta\gamma}^{[1]} = \partial\Psi^{[1]}/\partial\zeta, \qquad Eu_{\beta}^{[1]} = 2(1+\nu)\,\Psi^{[1]}$$

where the harmonic function $\Psi^{[1]}$ is represented by the series

$$\Psi^{[1]} = \sum_{n=1}^{\infty} \frac{16(-1)^n}{(2n-1)^3\pi^3} \exp\frac{(2n-1)\pi\xi}{2} \sin\frac{(2n-1)\pi\zeta}{2} \tag{2.6}$$

Then, as follows from the second condition $(1.16)_0$ and the Equation (1.5), the function $\Psi^{(0)}$ is determined by Equation

$$\Psi^{(0)} = \tau_{\alpha\beta}^{(0)}(\alpha_0,\ \beta)\,\Psi^{[1]} \tag{2.7}$$

Now we will consider the sequence of boundary conditions to be imposed. From Equation (2.1) and keeping (1.5) in mind, we obtain

$$\sigma_{\alpha\alpha}^{(0)} = 0 \qquad \text{for} \quad \alpha = \alpha_0 \tag{2.8}$$

Then as it follows from the first and third conditions in $(1.16)_0$ and from (1.11) that

$$\sigma_{\alpha\alpha(2)}^{(0)} = \sigma_{\alpha\gamma(2)}^{(0)} = 0 \qquad \text{for} \quad \xi = 0 \tag{2.9}$$

and hence

$$Q_{(2)}^{(0)} \equiv 0, \qquad \text{or} \quad \Phi^{(0)} \equiv 0 \tag{2.10}$$

The functions $B^{(0)}$ will be detrmined by (2.1) and (2.3), which, with the aid of (2.6), (2.7) and (2.10), can be represented in the form

$$\int_{-1}^{1} \zeta\sigma_{\alpha\alpha}^{(0)}\Big|_{\alpha=\alpha_0} d\zeta = 0, \qquad \int_{-1}^{1} \sigma_{\alpha\gamma}^{(0)}\Big|_{\alpha=\alpha_0} d\zeta + \frac{2}{3}\frac{\partial\tau_{\alpha\beta}^{(0)}}{\partial s_\beta}\Big|_{\alpha=\alpha_0} = 0 \quad \left(\frac{\partial}{\partial s_\beta} = H_\beta \frac{\partial}{\partial\beta}\right) \tag{2.11}$$

The solution of system $(1.7)_1$ will be written in the form

$$Q_{(1)}^{(1)} = \tau_{\alpha\beta}^{(1)} Q^{[1]} + Q_{(1)}^{*(1)} \tag{2.12}$$

where $Q^{[1]}$ is the solution of Problem 1, and $Q_{(1)}^{*(1)}$ is the particular solution of the nonhomogeneous system $(1.7)_1$ that satisfies the homogeneous conditions on $\xi = 0$.

The function $B^{(1)}$ will be determined by taking the boundary conditions (2.2) and (2.4) into account. Keeping (2.12), (2.6), (2.7), (2.10), $(1.7)_1$ and $(1.8)_1$ in mind, the above conditions can be written in the form

$$\int_{-1}^{1} \zeta\sigma_{\alpha\alpha}^{(1)}\Big|_{\alpha=\alpha_0} d\zeta = -\frac{2}{3} A \frac{\partial\tau_{\alpha\beta}^{(0)}}{\partial s_\beta}\Big|_{\alpha=\alpha_0}, \qquad A = \frac{384}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5} \approx 1.26009 \tag{2.13}$$

$$\int_{-1}^{1} \sigma_{\alpha\gamma}^{(1)}\Big|_{\alpha=\alpha_0} d\zeta + \frac{2}{3}\frac{\partial\tau_{\alpha\beta}^{(1)}}{\partial s_\beta}\Big|_{\alpha=\alpha_0} = -\frac{2}{3} A \frac{\partial\tau_\alpha}{\partial s_\beta} k_{\beta 0}\tau_{\alpha\beta}^{(0)}\Big|_{\alpha=\alpha_0}$$

Then we determine $\psi^{(1)}$ by Formula (2.12) and $\Phi^{(1)}$ by taking boundary conditions $(1.16)_1$ into account. It is easily seen that the process of determining the functions is recursive in nature, i.e. all quantities required at the nth stage are found from the preceding stages.

When the edge is simply supported, we consider the boundary conditions $(1.17)_0$, $(1.17)_1$ and the third equation in $(1.17)_2$ which can be written in the form

$$w^{(2)} - \frac{\nu}{2E}\zeta^2(\tau_{\alpha\alpha}^{(0)} + \tau_{\beta\beta}^{(0)}) + W_{(2)}^{(0)} = 0 \qquad \text{for } \alpha = \alpha_0\,(\xi = 0) \tag{2.14}$$

and the damping conditions (2.1) and (2.2). Instead of Equation (2.3) we will here use the damping condition obtained in [6]

$$\frac{2-\nu}{3}\int_{-1}^{1}\zeta^3\sigma_{\alpha\alpha(2)}^{(0)}\Big|_{\xi=0} d\zeta - \frac{E}{1+\nu}\int_{-1}^{1}(\zeta^2-1)W_{(2)}^{(0)}\Big|_{\xi=0} d\zeta = 0 \tag{2.15}$$

We will eliminate the quantity $w^{(2)}$ from (2.14) by means of (2.15) and the fact that $\sigma_{\alpha\alpha}^{(0)}|_{\alpha=\alpha_0} = 0$. as a consequence of (2.1) and (1.15). Then from (2.14) and the first equation in $(1.16)_0$ we obtain the boundary conditions for $\Phi^{(0)}$ in the form

$$\sigma_{\alpha\alpha(2)}^{(0)} = 0, \qquad W_{(2)}^{(0)} = -\frac{\nu}{10E}\tau_{\beta\beta}^{(0)}(1-5\zeta^2) \qquad \text{for } \xi = 0\,(\alpha = \alpha_0) \tag{2.16}$$

Let us now consider the auxiliary Problem 2. We will construct the solution $Q^{[2]}$ of the homogeneous system $(1.8)_0$ in the half-strip $-1 \leqslant \zeta \leqslant 1$, $\xi \leqslant 0$ which satisfies the conditions

$$\sigma_{\alpha\gamma}^{[2]}\Big|_{\zeta=\pm 1} = \sigma_{\gamma\gamma}^{[2]}\Big|_{\zeta=\pm 1} = 0, \quad Q^{[2]}\Big|_{\xi=-\infty} = 0, \quad \sigma_{\alpha\alpha}^{[2]}\Big|_{\xi=0} = 0, \quad EW^{[2]}\Big|_{\xi=0} = 1-5\zeta^2$$

Then the quantities $Q^{(0)}{}_{(2)}$ (the function $\Phi^{(0)}$) is determined by Formula

$$Q_{(2)}^{(0)} = -\frac{\nu}{10}\tau_{\beta\beta}^{(0)}(\alpha_0, \beta)\, Q^{[2]} \tag{2.17}$$

Now we will persue the successive determinations of the functions $B^{(0)}$, $B^{(1)}$, $\psi^{(0)}$ and $\Phi^{(0)}$. The function $B^{(0)}$ will be determined by (2.1) and the third equation in $(1.17)_0$ which has the form

$$w^{(0)} = 0, \qquad \int_{-1}^{1}\zeta\sigma_{\alpha\alpha}^{(0)} d\zeta = 0 \qquad \text{for } \alpha = \alpha_0 \tag{2.18}$$

Then we determine $\Psi^{(0)}$ by Formula (2.7) and $\Phi^{(0)}$ by Formula (2.17). The function $B^{(1)}$ can be determined from the third condition in $(1.17)_1$ and condition (2.2), which is easily put in the form

$$w^{(1)}=0, \qquad \int_{-1}^{1}\zeta\sigma_{\alpha\alpha}^{(1)}\,d\zeta=-\frac{2}{3}A\,\frac{\partial\tau_{\alpha\beta}^{(0)}}{\partial s_\beta}-\frac{2}{3}k_{\beta 0}B\,\tau_{\beta\beta}^{(0)} \qquad \text{for } \alpha=\alpha_0 \tag{2.19}$$

$$\left(B=\frac{3\nu^2}{40}\int_{-1}^{1}\zeta^2\sigma_{\alpha\gamma}^{[2]}\Big|_{\xi=0}d\zeta\right)$$

It is easily seen that in this case the process of determining the boundary conditions is recursive. When the edge is rigidly fixed, the determination of $B^{(0)}$, $\Psi^{(0)}$, $\Phi^{(0)}$ and $B^{(1)}$ is achieved by using conditions $(1.18)_0$, $(1.18)_1$, (2.14) and the damping condition $(1.14)_0$ which by virtue of (1.9) and (1.11) can be written in the form

$$\int_{-1}^{1}\zeta\sigma_{\alpha\alpha(2)}^{(0)}\Big|_{\xi=0}d\zeta=0, \qquad \int_{-1}^{1}\sigma_{\alpha\gamma(2)}^{(0)}\Big|_{\xi=0}d\zeta=0 \tag{2.20}$$

Now we will treat the auxiliary Problems 3,4 and 5. We will construct the solution of the homogeneous system $(1.8)_0$ in the half-strip $-1\leqslant\zeta\leqslant 1$, $-L\leqslant\xi\leqslant 0$ (L is a sufficiently large number) subject to conditions

$$\sigma_{\alpha\gamma}\big|_{\zeta=\pm 1}=\sigma_{\gamma\gamma}\big|_{\zeta=\pm 1}=0, \qquad u_\alpha\big|_{\xi=-\infty}=W\big|_{\xi=-\infty}=0$$

$u_\alpha\big|_{\xi=0}=0, \quad W\big|_{\xi=0}=1$ for problem 3

$u_\alpha\big|_{\xi=0}=0, \quad W\big|_{\xi=0}=\zeta^2$ for problem 4

$u_\alpha\big|_{\xi=0}=\zeta, \quad W\big|_{\xi=0}=0$ for problem 5

The solutions of Problems 3, 4 and 5 will be denoted by $Q^{[3]}$, $Q^{[4]}$ and $Q^{[5]}$, respectively.

Then, by virtue of (2.14), (1.5) and the first equation in $(1.18)_1$ we have

$$Q_{(2)}^{(0)}=-w^{(2)}(\alpha_0,\beta)\,Q^{[3]}+\frac{\nu}{2E}\left[\tau_{\alpha\alpha}^{(0)}(\alpha_0,\beta)+\tau_{\beta\beta}^{(0)}(\alpha_0,\beta)\right]Q^{[4]}-v_\alpha^{(1)}(\alpha_0,\beta)\,Q^{[5]} \tag{2.21}$$

Substituting (2.21) into (2.20) and solving the resulting equations for $w^{(2)}$ and $v^{(1)}_\alpha$, we have

$$w^{(2)}=K\frac{\nu}{2E}(\tau_{\alpha\alpha}^{(0)}+\tau_{\beta\beta}^{(0)}), \qquad v_\alpha^{(1)}=C\frac{\nu}{2E}(\tau_{\alpha\alpha}^{(0)}+\tau_{\beta\beta}^{(0)}) \qquad \text{for } \alpha=\alpha_0 \tag{2.22}$$

where the constants C and K can easily be determined if the solutions of Problems 3, 4 and 5 have been found.

Now we will persue the successive determinations of $B^{(0)}$, $B^{(1)}$, $\Psi^{(0)}$ and $\Phi^{(0)}$ The function $B^{(0)}$ is determined by the first and third conditions in $(1.18)_0$ which have the form

$$w^{(0)}=0, \qquad u_\alpha^{(0)}=0 \qquad \text{for } \alpha=\alpha_0 \tag{2.23}$$

(The second condition in $(1.18)_0$ is satisfied identically). We will determine the function $\Psi^{(0)}$ with the aid of the second condition in $(1.18)_1$. It is evident that $u^{(0)}_{\beta(1)}|_{\xi=0}=0$ by virtue of the third equation in $(1.18)_1$ and thus $\Psi^{(0)}\equiv 0$. Now we will determine $B^{(1)}$ with the aid of the third equation in $(1.18)_1$ and the second condition in (2.22), i.e. the conditions

$$w^{(1)}=0, \qquad v_\alpha^{(1)}=C\frac{\nu}{2E}(\tau_{\alpha\alpha}^{(0)}+\tau_{\beta\beta}^{(0)}) \qquad \text{for } \alpha=\alpha_0 \tag{2.24}$$

The function $\Phi^{(0)}$ will be determined by means of (2.14) and the first equation in $(1.18)_1$. Moreover, $\Phi^{(0)}$ can be expressed in terms of the solutions of the Problems 3, 4 and 5 by means of Formula (2.21).

3. Now we will turn to the question of constructing approximate methods of refining the classic bending theory of plates.

From the viewpoint of asymptotic methods, the problem of constructing various approximate theories of bending (and extension) of plates can be regarded as a problem of constructing a certain number of approximations in the basic and the auxiliary iteration processes. In [1 and 3 to 5] it was shown that, in the case of a straight edge or a curved free edge, the classical theory is equivalent to the problem of constructing the zeroth approximation of the basic iteration process. Below, it has been shown that this is also valid for the present variants in the case of an arbitrary smooth curvilinear boundary.

As is well known [1 and 2], the equations of the zeroth approximation in the basic iteration process are identical with the equations of the classical bending theory of plates. It is not difficult to verify that the boundary conditions for $B^{(0)}$ are also identical with the boundary conditions of the classical theory.

For $i = 0, 1$ we will employ the following expressions introduced in [2]:

$$M_\alpha^{(i)} = h^i \int_{-1}^{1} \zeta \sigma_{\alpha\alpha}^{(i)} d\zeta \quad (\alpha\beta), \qquad H_{\alpha\beta}^{(i)} = h^i \int_{-1}^{1} \zeta \sigma_{\alpha\beta}^{(i)} d\zeta$$

$$N_\alpha^{(i)} = h^i \int_{-1}^{1} \sigma_{\alpha\gamma}^{(i)} d\zeta \quad (\alpha\beta), \qquad v_\alpha^{(i)} = H_\alpha \frac{\partial w^{(i)}}{\partial \alpha} = \frac{\partial w^{(i)}}{\partial s_\alpha} \quad (\alpha\beta)$$

$$w_0^{(i)} = h^{i-3} w^{(i)}$$

Then, keeping (1.5) in mind, Expressions (2.11), (2.18) and (2.23) lead simply to the usual boundary conditions of the classical theory

for a free edge

$$M_\alpha^{(0)} = 0, \quad N_\alpha^{(0)} + \frac{\partial H_{\alpha\beta}^{(0)}}{\partial s_\beta} = 0 \quad \text{for } \alpha = \alpha_0$$

for a simply supported edge

$$M_\alpha^{(0)} = 0, \quad w_0^{(0)} = 0 \quad \text{for } \alpha = \alpha_0$$

for a fully fixed edge

$$w_0^{(0)} = 0, \quad \frac{\partial w_0^{(0)}}{\partial s_\alpha} = 0 \quad \text{for } \alpha = \alpha_0$$

It should be pointed out that in contrast to the classical theory the zeroth approximation in the basic iteration process can also be used to determine the stress $\sigma_{\gamma\gamma}$ (it may be substantial, for example, in certain anisotropic materials).

The remaining approximations in the basic iteration process and all approximations in the auxiliary iteration process give the corrections to the classical theory. Therefore, by limiting oneself to various numbers of approximations of the basic and auxiliary iterative processes, one can construct various approximate theories which with the corresponding degrees of accuracy will decrease the error in the classical theory either at the edge or at a distance from the edge. Of all these theories, we will only consider two in this paper. The first makes it possible to refine the results of the classical theory near the edge of the plate, which is important for example in problems on the stress concentration near holes. The second theory refines the results at points distant from the edge.

As follows from (1.2) the order of the stresses and displacements [3] determined by the zeroth approximation in the basic iteration process will be

$$\sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\beta\beta} \sim h^{-2}, \quad \sigma_{\alpha\gamma}, \sigma_{\beta\gamma} \sim h^{-1}, \quad \sigma_{\gamma\gamma} \sim h^{\circ}, \quad u_\alpha, u_\beta \sim h^{-2}, \quad W \sim h^{-3} \quad (3.1)$$

Near the edge of the plate, boundary-layer stresses determined by the auxiliary iteration process will be superposed on the basic state of stress. In the auxiliary iteration process the quantities $\sigma_{\alpha\beta}, \sigma_{\beta\gamma}$ and u_β can be determined from the equations for the edge twisting $(\Psi^{(s)})$, and the quantities $\sigma_{\alpha\alpha}, \sigma_{\alpha\gamma}, \sigma_{\beta\beta}, \sigma_{\gamma\gamma}, u_\alpha$ and W are determined from the equations for the plane deformation ($\Phi^{(s)}$), at the edge. Moreover, it follows from Section 2 that, depending on boundary conditions on the edge with $s = 0$, one (but never both simultaneously) of the states of stress at the edge can be absent. As is found from (1.6), the orders of the stresses and displacements for the zeroth approximation in the auxiliary iteration process will be

$$\sigma_{\alpha\alpha},\ \sigma_{\alpha\beta},\ \sigma_{\alpha\gamma},\ \sigma_{\beta\beta},\ \sigma_{\beta\gamma},\ \sigma_{\gamma\gamma} \sim h^{-2}, \qquad u_\alpha,\ u_\beta, \qquad W \sim h^{-1} \tag{3.2}$$

It follows from (3.1) and (3.2) that, for any of the considered variants of boundary conditions, the error allowed by the classical theory of the edge of the plate will be small in the case of the displacements. However, in the case of the stresses (all or some) the error will be of the same order as in the basic stresses in the classical theory. Thus, when it is very important to determine the stresses at the edge of the plate, it is insufficient to limit oneself to the classical theory for the construction of the state of stress even in the zeroth approximation.

The asymptotic method makes it possible to very simply formulate an approximate theory that allows the construction of the zeroth approximation for the state of stress either removed from the edge or at the edge. It is easy to see that the essence of the method is that one superposes boundary-layer stresses determined by the zeroth approximation in the auxiliary iteration process on the state of stress determined by the classical theory. Using the notation $B^{(s)}$, $\Psi^{(s)}$, $\Phi^{(s)}$, the problem of setting up such a theory can be formulated as the problem of constructing the following functions: $B^{(0)}$ and $\Psi^{(0)}$ for the free edge, $B^{(0)}$, $\Psi^{(0)}$ and $\Phi^{(0)}$ for the simply supported edge, or $B^{(0)}$ and $\Phi^{(0)}$ for the fully fixed edge. (The boundary conditions for these functions and the formulas with which one determines $\Psi^{(0)}$ and $\Phi^{(0)}$ were given in Section 2). The error in such a theory for the stress will everywhere have the order h compared with h^0.

The asymptotic method also enables one to very simply set up an approximate theory that makes the classical theory more exact only at a distance from the edge. Such a theory will comprise the first two approximations of the basic iterative process (functions $B^{(0)}$ and $B^{(1)}$). As has been shown earlier [1 to 3], the construction of the first approximation in the basic iterative processes ($B^{(1)}$) reduces to the solution of a homogeneous biharmonic equation (in $\alpha\beta$) with nonhomogeneous conditions which, in the terminology of the classical theory, can be represented in the form

for the free edge

$$M_\alpha^{(1)} = -Ah\frac{\partial H_{\alpha\beta}^{(0)}}{\partial s_\beta}, \qquad N_\alpha^{(1)} + \frac{\partial H_{\alpha\beta}^{(1)}}{\partial s_\beta} = -Ah\frac{\partial}{\partial s_\beta}\left(k_{\beta 0} H_{\alpha\beta}^{(0)}\right) \qquad \text{for } \alpha = \alpha_0$$

for the simply supported edge

$$w_0^{(1)} = 0, \qquad M_\alpha^{(1)} = -hA\frac{\partial H_{\alpha\beta}^{(0)}}{\partial s_\beta} - Bk_{\beta 0} h M_\beta^{(0)} \qquad \text{for } \alpha = \alpha_0$$

for the fully fixed edge

$$w_0^{(1)} = 0, \qquad \frac{\partial w_0^{(1)}}{\partial s_\alpha} = C\frac{\nu}{2D(1-\nu^2)}h\left(M_\alpha^{(0)} + M_\beta^{(0)}\right) \qquad \text{for } \alpha = \alpha^0 \qquad \left(D = \frac{2Eh^3}{3(1-\nu^2)}\right)$$

By combining $B^{(0)}$ and $B^{(1)}$, we obtain an approximate method of refining the classical bending theory for plates that will consist in the construction of a solution of the equations of the classical theory subject to the following modified boundary-value conditions (to an accuracy of terms of the order h^2 compared with h^0).

for the free edge

$$M_\alpha + Ah\frac{\partial H_{\alpha\beta}}{\partial s_\beta} = 0, \qquad N_\alpha + \frac{\partial}{\partial s}\left(H_{\alpha\beta} + Ahk_{\beta 0} H_{\alpha\beta}\right) = 0 \qquad \text{for } \alpha = \alpha_0$$

for the simply supported edge

$$M_\alpha + Ah\frac{\partial H_{\alpha\beta}}{\partial s_\beta} + Bk_{\beta 0} h M_\beta = 0, \qquad w_0 = 0 \qquad \text{for } \alpha = \alpha_0$$

for the rigidly fixed edge

$$w_0 = 0, \quad \frac{\partial w_0}{\partial s_\alpha} - C\frac{\nu}{2D(1-\nu^2)} h(M_\alpha + M_\beta) = 0 \qquad \text{for } \alpha = \alpha_0$$

The error of such a theory will have the order h^2 compared with h^0. The attempt to further refine the classical theory (determination of errors of order h^2 compared to h^0) leads to the necessity of adding corrections both to the equation of the classical theory and to the associated boundary conditions [2 and 3].

4. In the symmetric problem the constant of integration in the equations for $B^{(s)}$ $\Psi^{(s)}$ and $\Phi^{(s)}$ must be used in order to satisfy the boundary conditions (1.16) to (1.18) and the damping conditions (1.15). Without dwelling on the transformations which are analogous to the transformations for the antisymmetric problem, we will formulate approximate methods of refining the classical theory at the edge and at a distance from it.

In the case of a free edge, the classical theory (the zeroth approximation in the basic iteration process) has at a distance from the edge an error in all stresses of the order h^2 compared with h^0. At the edge the error in $\sigma_{\alpha\beta}$ and $\sigma_{\beta\gamma}$ is of the order h^2 and for $\sigma_{\alpha\alpha}, \sigma_{\alpha\gamma}, \sigma_{\beta\beta}$ and $\sigma_{\gamma\gamma}$ the error is of the order h compared with h^0.

Therefore, when the plate has free edges the classical theory enables one in the zeroth approximation to construct states of stress either at a distance rom the edge or at the edge. If the edge is simply supported, then at a distance from it the error in the calssical theory in all stresses is of the order h, and at the edge the error in $\sigma_{\alpha\alpha}, \sigma_{\alpha\gamma}, \sigma_{\beta\beta}$ and $\sigma_{\gamma\gamma}$ is of the order h^0, and in $\sigma_{\alpha\beta}$ and $\sigma_{\beta\gamma}$ it is of the order h compared with h^0.

The approximate method, which makes it possible to construct in the zeroth approximation the state of stress not only at a distance from but also at the edge, consists in the construction of functions $B^{(0)}$ and $\Phi^{(0)}$ from the following boundary condition

for $B^{(0)}$

$$\tau_{\alpha\alpha}^{(0)} = \tau_{\alpha\beta}^{(0)} = 0 \qquad \text{for } \alpha = \alpha_0$$

for Φ^0

$$\sigma_{\alpha\alpha(2)}^{(0)} = 0, \quad EW_{(2)}^{(0)} = \nu\zeta\tau_{\beta\beta}^{(0)} \qquad \text{for } \xi = 0 \ (\alpha = \alpha_0)$$

The approximate theory that allows the refinement of the classical theory only at a distance from the edge consists in the construction of a solution of the equations of the classical theory subject to the conditions

$$\tau_{\alpha\alpha} - hA_1\nu k_{\beta 0}\tau_{\beta\beta} = 0, \quad \tau_{\alpha\beta} + hA_1\nu\frac{\partial\tau_{\beta\beta}}{\partial s_\beta} = 0 \qquad \text{for } \alpha = \alpha_0 \quad \left(A_1 = \frac{\nu}{2}\int_{-1}^{1}\zeta\sigma_{\alpha\gamma}^{[6]}\Big|_{\xi=0} d\zeta\right)$$

where $\sigma_{\alpha\gamma}^{[6]}$ is the solution of the homogeneous system $(1.8)_0$ subject to the conditions

$$\sigma_{\alpha\gamma}^{[6]}\Big|_{\zeta=\pm1} = \sigma_{\gamma\gamma}^{[6]}\Big|_{\zeta=\pm1} = 0, \qquad Q^{[6]}\Big|_{\xi=-\infty} = 0, \qquad \sigma_{\alpha\alpha}^{[6]}\Big|_{\xi=0} = 0, \qquad W^{[6]}\Big|_{\xi=0} = -\zeta$$

The error in such a theory at a distance from the edge has the order h^2 compared with h^0.

In the case of a fully fixed edge, the construction in the zeroth approximation of the state of stress at a distance from the edge and at the edge necessitates the determination of the functions $B^{(0)}$ and $\Phi^{(0)}$ subject to the following boundary conditions

for $B^{(0)}$

$$v_\alpha^{(0)} = v_\beta^{(0)} = 0 \qquad \text{for } \alpha = \alpha_0$$

for $\Phi^{(0)}$

$$Eu_{\alpha(2)}^{(0)} = -B_1\nu(\tau_{\alpha\alpha}^{(0)} + \tau_{\beta\beta}^{(0)}), \qquad EW_{(2)}^{(0)} = \nu\zeta(\tau_{\alpha\alpha}^{(0)} + \tau_{\beta\beta}^{(0)}) \qquad \text{for } \xi = 0$$

In order to refine the data of the classical theory at a distance from the edge (the derivation of corrections of order h compared with h°) one must construct a solution of the equations of the classical theory subject to the following modified boundary conditions

$$v_\beta = 0, \quad Ev_\alpha - hB_1\nu(\tau_{\alpha\alpha} + \tau_{\beta\beta}) = 0 \quad \text{for } \alpha = \alpha_0$$

$$\left(B_1 = \int_{-1}^{1} \sigma_{\alpha\alpha}^{[7]}\Big|_{\xi=0} d\zeta \left(\int_{-1}^{1} \sigma_{\alpha\alpha}^{[8]}\Big|_{\xi=0} d\zeta\right)^{-1}\right)$$

and $Q^{[7]}$ and $Q^{[8]}$ are solutions of system $(1.8)_0$ with the conditions

$$\sigma_{\alpha\gamma}\big|_{\zeta=\pm 1} = \sigma_{\gamma\gamma}\big|_{\xi=\pm 1} = 0, \quad Q\big|_{\xi=-\infty} = 0$$

$$W\big|_{\xi=0} = -\zeta, \quad u_\alpha\big|_{\zeta=0} = 0 \quad \text{for Problem 7}$$

$$W\big|_{\xi=0} = 0, \quad u_\alpha\big|_{\xi=0} = 1 \quad \text{for Problem 8}$$

5. The above-considered symmetric problem is the problem of the extension of a plate by forces applied to its faces. Now we will consider the physically more interesting problem of the extension of a plate by forces applied to its edge surfaces.

We will assume that the faces of the plate are stress-free

$$\sigma_{\alpha\gamma} = \sigma_{\beta\gamma} = \sigma_{\gamma\gamma} = 0 \quad \text{for } \zeta = \pm 1$$

and the conditions on the edge $\alpha = \alpha_0$ ($\xi = 0$) have the form

$$\sigma_{\alpha\alpha} = a(\alpha_0, \beta, \zeta), \quad \sigma_{\alpha\beta} = b(\alpha_0, \beta, \zeta), \quad \sigma_{\alpha\gamma} = c(\alpha_0, \beta, \zeta) \quad \text{for } \alpha = \alpha_0$$

It is assumed that a and b are even functions of ζ and c is odd, and that they do not depend on h. The generalization to the case where these functions are proportional to some power of h or can be represented in the form of polynomials of h does not present difficulties for the linear problem.

It can be shown that in the present case one must choose the following values for q_1 and q_2

$$q_1 = 0 \quad \text{for } \sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\beta\beta}, \quad q_1 = -1 \quad \text{for } \sigma_{\alpha\gamma}, \sigma_{\beta\gamma}$$

$$q_1 = -2 \quad \text{for } \sigma_{\gamma\gamma}, \quad q_1 = 0 \quad \text{for } u_\alpha, u_\beta, \quad q_1 = -1 \quad \text{for } W$$

$$q_2 = 0 \quad \text{for } \sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\alpha\gamma}, \sigma_{\beta\beta}, \sigma_{\beta\gamma}, \sigma_{\gamma\gamma}$$

$$q_2 = -1 \quad \text{for } u_\alpha, u_\beta, W$$

The system of equations for the basic state of stress comprises the usual equations of the plane problem and when $s = 0, 1$ can be reduced to a homogeneous biharmonic equation. The equations for the boundary-layer stresses have the form (1.7) and (1.8). The boundary conditions for the coefficients of the expansion (1.2) are obtained with the aid of a procedure written in [1 and 2] in the form

$$\sigma_{\alpha\alpha}^{(s)} + \sigma_{\alpha\alpha\,(1)}^{(s)} + \sigma_{\alpha\alpha\,(2)}^{(s)} = a^{(s)}, \quad \sigma_{\alpha\beta}^{(s)} + \sigma_{\alpha\beta\,(1)}^{(s)} + \sigma_{\alpha\beta\,(2)}^{(s)} = b^{(s)}$$

$$\sigma_{\alpha\gamma}^{(s-1)} + \sigma_{\alpha\gamma\,(1)}^{(s)} + \sigma_{\alpha\gamma\,(2)}^{(s)} = c^{(s)} \quad \text{for } \alpha = \alpha_0$$

$$(a^{(0)} = a, \quad b^{(0)} = b, \quad c^{(0)} = c, \quad a^{(k)} = b^{(k)} = c^{(k)} \equiv 0) \qquad (k > 0)$$

Adding here the damping conditions (1.15) we obtain five sequences of boundary conditions on $\alpha = \alpha_0$ for the functions $B^{(s)}$ $\Psi^{(s)}$ and $\Phi^{(s)}$

The approximate theory that enables one to refine the results of the classical theory near the edge of the plate consists in the construction of functions $B^{(0)}$ $\Psi^{(0)}$ and $\Phi^{(0)}$ subject to the following boundary conditions:

for $B^{(0)}$ $\quad \tau_{\alpha\alpha}^{(0)} = \frac{1}{2}\int_{-1}^{1} a\, d\zeta, \quad \tau_{\alpha\beta}^{(0)} = \frac{1}{2}\int_{-1}^{1} b\, d\zeta$ for $\alpha = \alpha_0$

for $\Psi^{(0)}$ $\quad \sigma_{\alpha\beta(1)}^{(0)} = b - \frac{1}{2}\int_{-1}^{1} b\, d\zeta$ for $\xi = 0$

for $\Phi^{(0)}$ $\quad \sigma_{\alpha\alpha(2)}^{(0)} = a - \frac{1}{2}\int_{-1}^{1} a\, d\zeta, \quad \sigma_{\alpha\gamma(2)}^{(0)} = c$ for $\xi = 0$

The error of such a theory for the stresses everywhere has order h compared with h^0.

It should be noted that in the case when a and b do not depend on ζ and $c \equiv 0$, the function $\Psi^{(0)}$ and $\Phi^{(0)}$ are identically zero and the state of stress in the zeroth approximation far from the edge as well as on the boundary can be constructed with the aid of the classical theory.

The approximate theory that allows one to refine the classical theory only at a distance from the edge consistes in the sllution of the (homogeneous) system of equations of the plane problem with the following boundary conditions on $\alpha = \alpha_0$:

$$\tau_{\alpha\alpha} = \frac{1}{2}\int_{-1}^{1} [a - hk_{\beta 0}\zeta c]\, d\zeta, \qquad \tau_{\alpha\beta} = \frac{1}{2}\int_{-1}^{1}\left[b - h\nu\zeta\frac{\partial c}{\partial s_\beta}\right] d\zeta$$

The error in this theory at a distance from the edge has order h^2 compared with h^0. If $c \equiv 0$, this theory coincides with the classical theory.

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